2016/17 MATH2230B/C Complex Variables with Applications
Problems in HW 4
Due Date on 6 Apr 2017

All the problems are from the textbook, Complex Variables and Application (9*th* edition).

1 P.237

1. Find the residue at $z = 0$ of the function
   (a) $\dfrac{1}{z + z^2}$;
   (b) $z \cos\left(\dfrac{1}{z}\right)$;
   (c) $\dfrac{z - \sin z}{z}$;
   (d) $\dfrac{\cot z}{z^4}$;
   (e) $\dfrac{\sinh z}{z^4(1 - z^2)}$.

2. Use Cauchy's residue theorem (Sec. 76) to evaluate the integral of each of these functions around the circle $|z| = 3$ in the positive sense:
   (a) $\dfrac{\exp(-z)}{z^2}$;
   (b) $\dfrac{\exp(-z)}{(z - 1)^2}$;
   (c) $z^2 \exp\left(\dfrac{1}{z}\right)$;
   (d) $\dfrac{z + 1}{z^2 - 2z}$.

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4. Use the theorem in Sec. 77, involving a single residue, to evaluate the integral of each of these functions around the circle $|z| = 2$ in the positive sense:
   (a) $\dfrac{z^5}{1 - z^3}$;
   (b) $\dfrac{1}{1 + z^2}$;
   (c) $\dfrac{1}{z}$.

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1. In each case, write the principal part of the function at its isolated singular point and determine whether that point is a removable singular point, an essential singular point, or a pole:

   (a) $z \exp\left(\frac{1}{z}\right)$;

   (b) $\frac{z^2}{1+z}$;

   (c) $\frac{\sin z}{z}$;

   (d) $\frac{\cos z}{z}$;

   (e) $\frac{1}{(2-z)^3}$.

2. Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue B.

   (a) $\frac{1-\cosh z}{z^3}$;

   (b) $\frac{1-\exp(2z)}{z^4}$;

   (c) $\frac{\exp(2z)}{(z-1)^2}$.

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3. In each case, find the order m of the pole and the corresponding residue B at the singularity $z = 0$:

   (a) $\frac{\sinh z}{z^4}$;

   (b) $\frac{1}{z(e^z - 1)}$.

4. Find the value of the integral

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\,\mathrm{d}z,$$

   taken counterclockwise around the circle (a) $|z-2| = 2$; (b) $|z| = 4$.

5. Find the value of the integral

$$\int_C \frac{\mathrm{d}z}{z^3(z+4)},$$

   taken counterclockwise around the circle (a) $|z| = 2$; (b) $|z+2| = 3$.

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5. Let C denote the positively oriented circle $|z| = 2$ and evaluate the integral

   (a) $\int_C \tan z \, \mathrm{d}z$;

   (b) $\int_C \frac{\mathrm{d}z}{\sinh 2z}$.

7. Show that

$$\int_C \frac{\mathrm{d}z}{(z^2-1)^3+3} = \frac{\pi}{2\sqrt{2}},$$

   where C is the positvely oriented boundary of the rectangle whose sides lie along the lines $x = \pm 2$, $y = 0$ and $y = 1$.
   Suggestion: By observing that the four zeros of the polynomial $q(z) = (z^2-1)^3+3$ are the square roots of the numbers $1 + \pm\sqrt{3}i$, show that the reciprocal $1/q(z)$ is analytic inside and on C except at the points

$$z_0 = \frac{\sqrt{3}+i}{\sqrt{2}} \quad \text{and} \quad -\overline{z_0} = \frac{-\sqrt{3}+i}{\sqrt{2}}.$$

   Then apply Theorem 2 in Sec. 83.

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Use residues to derive the integration formulas in 2. and 4.

2. $\int_0^\infty \frac{\mathrm{d}x}{(x^2+1)^2} = \frac{\pi}{4}$.

4. $\int_0^\infty \frac{x^2 \, \mathrm{d}x}{x^6+1} = \frac{\pi}{6}$.

9. Use a residue and the contour

$$C = [0, R] \cup \{Re^{i\theta} : 0 \le \theta \le 2\pi/3\} \cup \{re^{i2\pi/3} : 0 \le r \le R\} \quad \text{(counterclockwise)}$$

   where $R > 1$, to establish the integration formula

$$\int_0^\infty \frac{\mathrm{d}x}{x^3+1} = \frac{2\pi}{3\sqrt{3}}.$$